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Author: Karol Baron

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Aequationes Mathematicae

Weak law of large numbers for iterates of random-valued functions

Karol Baron

To the memory of Professor Marek Kuczma and Professor György Targoński.

Abstract. Given a probability space $(\Omega, \mathcal{A}, P)$, a complete and separable metric space X with the σ-algebra $\mathcal{B}$ of all its Borel subsets and a $\mathcal{B} \otimes \mathcal{A}$-measurable $f : X \times \Omega \to X$ we consider its iterates f^n defined on $X \times \Omega^{\mathbb{N}}$ by $f^0(x, \omega) = x$ and $f^n(x, \omega) = f\big(f^{n-1}(x, \omega), \omega_n\big)$ for $n \in \mathbb{N}$ and provide a simple criterion for the existence of a probability Borel measure π on X such that for every $x \in X$ and for every Lipschitz and bounded $\psi : X \to \mathbb{R}$ the sequence $\left(\frac{1}{n} \sum_{k=0}^{n-1} \psi\big(f^k(x, \cdot)\big)\right)_{n \in \mathbb{N}}$ converges in probability to $\int_X \psi(y)\pi(dy)$.

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1. Introduction

Fix a probability space $(\Omega, \mathcal{A}, P)$ and a complete and separable metric space (X, ρ).

Let $\mathcal{B}$ denote the σ-algebra of all Borel subsets of X. We say that $f : X \times \Omega \to X$ is a *random-valued* function (shortly: an *rv-function*) if it is measurable with respect to the product σ-algebra $\mathcal{B} \otimes \mathcal{A}$. The iterates of such an rv-function are given by

$$f^0(x, \omega_1, \omega_2, \dots) = x, \quad f^n(x, \omega_1, \omega_2, \dots) = f\big(f^{n-1}(x, \omega_1, \omega_2, \dots), \omega_n\big)$$

for $n \in \mathbb{N}$, $x \in X$ and $(\omega_1, \omega_2, \dots)$ from Ω^∞ defined as $\Omega^{\mathbb{N}}$. Note that $f^n : X \times \Omega^\infty \to X$ is an rv-function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. More exactly, for $n \in \mathbb{N}$ the nth iterate f^n is $\mathcal{B} \otimes \mathcal{A}_n$-measurable, where $\mathcal{A}_n$

denotes the σ-algebra of all sets of the form

$$\{(\omega_1, \omega_2, \dots) \in \Omega^\infty : (\omega_1, \dots, \omega_n) \in A\}$$

with A from the product σ-algebra $\mathcal{A}^n$. (See [4], [5, Sec. 1.4].)

A result on the a.s. convergence of $\big(f^n(x,\cdot)\big)_{n\in\mathbb{N}}$ for X being the unit interval may be found in [5, Sec. 1.4B]. The paper [4] by Rafał Kapica brings theorems on the convergence a.s. and in L^1 of those sequences of iterates in the case where X is a closed subset of a Banach lattice. A simple criterion for the convergence in law of $(f^n(x,\cdot))_{n\in\mathbb{N}}$ to a random variable independent of $x \in X$ was proved in [1] and applied to the equation

$$\varphi(x) = \int_\Omega \varphi\big(f(x,\omega)\big)P(d\omega) + F(x)$$

with φ as the unknown function. In [2] this criterion was applied to the equation

$$\varphi(x) = F(x) - \int_\Omega \varphi\big(f(x,\omega)\big)P(d\omega).$$

In the present paper it is strengthened and applied to get a weak law of large numbers for iterates of random-valued functions.

2. Wasserstein metric

By a distribution (on X) we mean any probability measure defined on $\mathcal{B}$. Recall that a sequence $(\pi_n)_{n\in\mathbb{N}}$ of distributions converges weakly to a distribution π if $\lim\limits_{n\to\infty} \int_X u(x)\pi_n(dx) = \int_X u(x)\pi(dx)$ for any continuous and bounded $u : X \to \mathbb{R}$. It is well known (see [3, Th. 11.3.3]) that this convergence is metrizable by the (Fortet–Mourier, Lévy–Prohorov, Wasserstein) metric

$$\|\pi_1 - \pi_2\|_W = \sup\left\{\left|\int_X u d\pi_1 - \int_X u d\pi_2\right| :\ u \in \mathrm{Lip}_1(X), \|u\|_\infty \le 1\right\},$$

where

$$\mathrm{Lip}_1(X) = \{u : X \to \mathbb{R} |\ |u(x) - u(z)| \le \varrho(x,z) \text{ for } x, z \in X\}$$

and $\|u\|_\infty = \sup\{|u(x)| :\ x \in X\}$ for a bounded $u : X \to \mathbb{R}$.

3. Convergence in law

Fix an rv-function $f : X \times \Omega \to X$ and let $\pi_n(x,\cdot)$ denote the distribution of $f^n(x,\cdot)$, i.e.,

$$\pi_n(x,B) = P^\infty\big(f^n(x,\cdot) \in B\big) \quad \text{for } n \in \mathbb{N}\cup\{0\},\ x \in X \text{ and } B \in \mathcal{B}.$$

The above mentioned strengthening of [1, Th. 3.1] reads as follows.

Theorem 3.1. *If*

$$\int_\Omega \varrho\big(f(x,\omega), f(z,\omega)\big)P(d\omega) \le \lambda\varrho(x,z) \quad \textit{for} \;\; x,z\in X \tag{1}$$

with a $\lambda\in(0,1)$, *and*

$$\int_\Omega \varrho\big(f(x,\omega), x\big)P(d\omega) < \infty \quad \textit{for} \;\; x\in X, \tag{2}$$

then there exists a distribution π *on* X *such that for every* $x\in X$ *the sequence* $\big(\pi_n(x,\cdot)\big)_{n\in\mathbb{N}}$ *converges weakly to* π*; moreover,*

$$\|\pi_n(x,\cdot)-\pi\|_W \le \frac{\lambda^n}{1-\lambda}\int_X \varrho\big(f(x,\omega),x\big)P(d\omega) \quad \textit{for} \;\; x\in X \;\; \textit{and} \;\; n\in\mathbb{N} \tag{3}$$

and

$$\int_X \varrho(x,y)\pi(dy) < \infty \quad \textit{for} \;\; x\in X. \tag{4}$$

Proof. It follows from [1, Th. 3.1] that there exists a distribution π on X such that (3) holds. We shall show that (4) is also satisfied. To this end note first that by (1) we have

$$\int_{\Omega^\infty} \varrho\big(f^n(x,\omega), f^n(z,\omega)\big)P^\infty(d\omega) \le \lambda^n\varrho(x,z) \quad \text{for } x,z\in X \text{ and } n\in\mathbb{N}. \tag{5}$$

Fix $x\in X$ and for every $n\in\mathbb{N}$ define $\tau_n:[0,\infty)\to[0,\infty)$ by

$$\tau_n(t)=\min\{t,n\}.$$

Since, by (3),

$$\begin{aligned}\left|\int_X \tau_n\big(\rho(x,y)\big)\pi_n(x,dy) - \int_X \tau_n\big(\rho(x,y)\big)\pi(dy)\right| &\le n\|\pi_n(x,\cdot)-\pi\|_W \\ &\le \frac{n\lambda^n}{1-\lambda}\int_X \varrho\big(f(x,\omega),x\big)P(d\omega)\end{aligned}$$

for $n\in\mathbb{N}$ and by the monotone convergence theorem

$$\int_X \rho(x,y)\pi(dy) = \lim_{n\to\infty}\int_X \tau_n\big(\rho(x,y)\big)\pi(dy),$$

it is enough to prove that the sequence $\big(\int_X \tau_n\big(\rho(x,y)\big)\pi_n(x,dy)\big)_{n\in\mathbb{N}}$, i.e., the sequence $\big(\int_{\Omega^\infty}\tau_n\left(\rho\big(x,f^n(x,\omega)\big)\right)P^\infty(d\omega)\big)_{n\in\mathbb{N}}$, is bounded.

To show it observe that for every $n\in\mathbb{N}$ and $(\omega_1,\omega_2,\ldots)\in\Omega^\infty$ we have

$$\begin{aligned}\tau_n\left(\rho\left(f^n(x,\omega_1,\omega_2,\ldots),x\right)\right) &\le \rho\big(f^n(x,\omega_1,\omega_2,\ldots),x\big) \\ &= \rho\left(f^{n-1}\big(f(x,\omega_1),\omega_2,\omega_3,\ldots\big),x\right) \\ &\le \sum_{k=1}^n \rho\left(f^{n-k}\big(f(x,\omega_k),\omega_{k+1},\omega_{k+2},\ldots\big), f^{n-k}(x,\omega_{k+1},\omega_{k+2},\ldots)\right)\end{aligned}$$

and for every $y \in X$ the value $f^n(y, \omega_1, \omega_2, \ldots)$ depends only on y and on $(\omega_1, \ldots, \omega_n)$. Hence, applying the Fubini theorem and (5), for every $n \in \mathbb{N}$ we get

$$\begin{aligned}
&\int_{\Omega^\infty} \tau_n \left(\rho\big(f^n(x,\omega), x\big)\right) P^\infty(d\omega) \\
&\quad \leq \sum_{k=1}^{n} \int_{\Omega^\infty} \rho\left(f^{n-k}\big(f(x,\omega_1), \omega_2, \omega_3, \ldots\big), f^{n-k}(x, \omega_2, \omega_3, \ldots)\right) P^\infty\big(d(\omega_1, \omega_2, \ldots)\big) \\
&\quad \leq \sum_{k=1}^{n} \lambda^{n-k} \int_{\Omega} \rho\big(f(x,\omega), x\big) P(d\omega) \leq \frac{1}{1-\lambda} \int_{\Omega} \varrho\big(f(x,\omega), x\big) P(d\omega).
\end{aligned}$$

□

Remark 3.2. If (1) holds with a $\lambda \in (0, \infty)$ and (2) is satisfied, then the function $\upsilon : X \to [0, \infty)$ defined by

$$\upsilon(x) = \int_{\Omega} \varrho\big(f(x,\omega), x\big) P(d\omega) \tag{6}$$

is Lipschitz.

4. Weak law of large numbers

Theorem 4.1. *If (1) holds with a $\lambda \in (0,1)$ and (2) is satisfied, then there exists a distribution π on X such that for every $x \in X$ and for every Lipschitz and bounded $\psi : X \to \mathbb{R}$ the sequence $\left(\frac{1}{n}\sum_{k=0}^{n-1} \psi \circ f^k(x, \cdot)\right)_{n \in \mathbb{N}}$ converges in probability to $\int_X \psi(y)\pi(dy)$.*

Proof. Making use of Theorem 3.1 let π be a distribution on X such that (3) and (4) hold. It follows from Remark 3.2 and (4) that

$$\int_X \upsilon(y)\pi(dy) < \infty. \tag{7}$$

Fix $x_0 \in X$, a Lipschitz and bounded $\psi : X \to \mathbb{R}$ and an $\epsilon \in (0, \infty)$. Put

$$\xi_n = \psi \circ f^n(x_0, \cdot) \quad \text{for} \quad n \in \mathbb{N}, \quad c = \int_X \psi(y)\pi(dy). \tag{8}$$

We shall show that

$$\lim_{n\to\infty} P^\infty\left(\left|\frac{1}{n}\sum_{k=0}^{n-1} \xi_k - c\right| \geq \epsilon\right) = 0.$$

Since by Chebyshev's inequality

$$P^\infty\left(\left|\frac{1}{n}\sum_{k=0}^{n-1}\xi_k - c\right| \ge \epsilon\right) \le \frac{1}{n^2\epsilon^2}\int_{\Omega^\infty}\left(\sum_{k=0}^{n-1}(\xi_k - c)\right)^2 dP^\infty \quad \text{for} \quad n\in\mathbb{N},$$

it is enough to prove that

$$\lim_{n\to\infty}\frac{1}{n^2}\int_{\Omega^\infty}\left(\sum_{k=0}^{n-1}(\xi_k - c)\right)^2 dP^\infty = 0.$$

We may assume that

$$\psi\in \mathrm{Lip}_1(X) \quad \text{and} \quad \|\psi\|_\infty \le 1. \tag{9}$$

We shall prove that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n-1}\sum_{l=0}^{k-1}\int_{\Omega^\infty}\xi_k\xi_l dP^\infty = \frac{c^2}{2}, \tag{10}$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=0}^{n-1}\int_{\Omega^\infty}\xi_k^2 dP^\infty = 0, \quad \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int_{\Omega^\infty}\xi_k dP^\infty = c. \tag{11}$$

Since

$$\int_{\Omega^\infty}\left(\sum_{k=0}^{n-1}(\xi_k - c)\right)^2 dP^\infty = 2\sum_{k=1}^{n-1}\sum_{l=0}^{k-1}\int_{\Omega^\infty}\xi_k\xi_l dP^\infty + \sum_{k=0}^{n-1}\int_{\Omega^\infty}\xi_k^2 dP^\infty - 2nc\sum_{k=0}^{n-1}\int_{\Omega^\infty}\xi_k dP^\infty + n^2c^2$$

for every integer $n\ge 2$, it will complete the proof.

Fix integers $n\ge 2$, $k\in[1,n-1]$ and $l\in[0,k-1]$. Then

$$f^k(x_0,\omega_1,\omega_2,\ldots) = f^{k-l}\big(f^l(x_0,\omega_1,\omega_2,\ldots),\omega_{l+1},\omega_{l+2},\ldots\big)$$

for $(\omega_1,\omega_2,\ldots)\in\Omega^\infty$. Hence, by (8) and the Fubini theorem,

$$\int_{\Omega^\infty}\xi_k\xi_l dP^\infty = \int_{\Omega^\infty}\left(\int_X \psi\big(f^{k-l}(y,\omega)\big)\psi(y)\pi_l(x_0,dy)\right)P^\infty(d\omega).$$

It follows from (9) and (5) that the function

$$x\mapsto \int_{\Omega^\infty}\psi\big(f^{k-l}(x,\omega)\big)P^\infty(d\omega), \quad x\in X,$$

has values in $[-1,1]$ and is Lipschitz with a Lipschitz constant λ^{k-l}, whence the function

$$x\mapsto \psi(x)\int_{\Omega^\infty}\psi\big(f^{k-l}(x,\omega)\big)P^\infty(d\omega), \quad x\in X,$$

has value in $[-1,1]$ and is Lipschitz with a Lipschitz constant $1+\lambda^{k-l}$. Hence and from (3) and (6) we infer that

$$\begin{aligned}
&\Big|\int_X \psi(y)\left(\int_{\Omega^\infty}\psi\big(f^{k-l}(y,\omega)\big)P^\infty(d\omega)\right)\pi_l(x_0,dy)\\
&\qquad -\int_X \psi(y)\left(\int_{\Omega^\infty}\psi\big(f^{k-l}(y,\omega)\big)P^\infty(d\omega)\right)\pi(dy)\Big|\\
&\quad \le 2\|\pi_l(x_0,\cdot)-\pi\|_W \le \frac{2\lambda^l}{1-\lambda}\upsilon(x_0).
\end{aligned}$$

Consequently, for every integer $n\ge 2$,

$$\begin{aligned}
&\left|\sum_{k=1}^{n-1}\sum_{l=0}^{k-1}\left(\int_{\Omega^\infty}\xi_k\xi_l dP^\infty-\int_X\psi(y)\big(\int_X\psi(z)\pi_{k-l}(y,dz)\big)\pi(dy)\right)\right|\\
&\quad\le \sum_{k=1}^{n-1}\sum_{l=0}^{k-1}\frac{2\lambda^l}{1-\lambda}\upsilon(x_0)=\frac{2\upsilon(x_0)}{1-\lambda}\sum_{k=1}^{n-1}\frac{1-\lambda^k}{1-\lambda}\le\frac{2(n-1)\upsilon(x_0)}{(1-\lambda)^2}.
\end{aligned}$$

It shows that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n-1}\sum_{l=0}^{k-1}\left(\int_{\Omega^\infty}\xi_k\xi_l dP^\infty-\int_X\psi(y)\big(\int_X\psi(z)\pi_{k-l}(y,dz)\big)\pi(dy)\right)=0. \tag{12}$$

Further, for every integer $n\ge 2$,

$$\begin{aligned}
&\sum_{k=1}^{n-1}\sum_{l=0}^{k-1}\int_X\psi(y)\big(\int_X\psi(z)\pi_{k-l}(y,dz)\big)\pi(dy)\\
&\qquad=\sum_{k=1}^{n-1}(n-k)\int_X\psi(y)\big(\int_X\psi(z)\pi_k(y,dz)\big)\pi(dy)
\end{aligned}$$

and, by (9), (3) and (6),

$$\left|\int_X\psi(z)\pi_k(y,dz)-\int_X\psi(z)\pi(dz)\right|\le\|\pi_k(y,\cdot)-\pi\|_W\le\frac{\lambda^k}{1-\lambda}\upsilon(y)$$

for $y \in X$ and $k \in \mathbb{N}$, whence

$$\begin{aligned}
&\Bigg|\sum_{k=1}^{n-1}(n-k)\int_X \psi(y)\Big(\int_X \psi(z)\pi_k(y,dz)\Big)\pi(dy) \\
&\quad -\sum_{k=1}^{n-1}(n-k)\int_X \psi(y)\Big(\int_X \psi(z)\pi(dz)\Big)\pi(dy)\Bigg| \\
&\le \sum_{k=1}^{n-1}(n-k)\int_X |\psi(y)|\left|\int_X \psi(z)\pi_k(y,dz)-\int_X \psi(z)\pi(dz)\right|\pi(dy) \\
&\le \sum_{k=1}^{n-1}(n-k)\int_X \frac{\lambda^k}{1-\lambda}\upsilon(y)\pi(dy) \le \frac{n-1}{1-\lambda}\int_X \upsilon(y)\pi(dy)\sum_{k=1}^{n-1}\lambda^k \\
&= \frac{(n-1)\lambda(1-\lambda^{n-1})}{(1-\lambda)^2}\int_X \upsilon(y)\pi(dy).
\end{aligned}$$

Since, by (8),

$$\sum_{k=1}^{n-1}(n-k)\int_X \psi(y)\Big(\int_X \psi(z)\pi(dz)\Big)\pi(dy) = \frac{n(n-1)}{2}c^2,$$

jointly with (7), it gives

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n-1}\sum_{l=0}^{k-1}\int_X \psi(y)\Big(\int_X \psi(z)\pi_{k-l}(y,dz)\Big)\pi(dy) = \frac{c^2}{2}.$$

Hence and from (12) we have (10).

From the weak convergence of $\big(\pi_n(x_0,\cdot)\big)_{n\in\mathbb{N}}$ to π it follows that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int_{\Omega^\infty}\xi_k dP^\infty = \int_X \psi(y)\pi(dy) = c$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int_{\Omega^\infty}\xi_k^2 dP^\infty = \int_X \psi(y)^2\pi(dy),$$

which shows that (11) also holds and ends the proof. □

Since continuous real functions defined on a compact metric space can be uniformly approximated by Lipschitz functions (see [3, Theorem 11.2.4]), Theorem 4.1 implies the following corollary.

Corollary 4.2. *Assume (X,ρ) is a compact metric space. If (1) holds with a $\lambda \in (0,1)$ and (2) is satisfied, then there exists a distribution π on X such that for every $x \in X$ and for every continuous $\psi : X \to \mathbb{R}$ the sequence $\big(\frac{1}{n}\sum_{k=0}^{n-1}\psi\circ f^k(x,\cdot)\big)_{n\in\mathbb{N}}$ converges in probability to $\int_X \psi(y)\pi(dy)$.*

Remark 4.3. In the results presented we cannot replace the sequence of means $\left(\frac{1}{n}\sum_{k=0}^{n-1}\psi\circ f^k(x,\cdot)\right)_{n\in\mathbb{N}}$ by $\left(\psi\circ f^n(x,\cdot)\right)_{n\in\mathbb{N}}$.

To see it fix a $\lambda\in(0,1)$ and an $\mathcal{A}$-measurable $\xi:\Omega\to[0,1-\lambda]$, and consider the rv-function $f:[0,1]\times\Omega\to[0,1]$ given by

$$f(x,\omega)=\lambda x+\xi(\omega).$$

We shall show that if $\left(\psi\circ f^n(x,\cdot)\right)_{n\in\mathbb{N}}$ converges in probability for an $x\in[0,1]$ and for a Borel $\psi:[0,1]\to\mathbb{R}$ such that

$$c|x-z|\le|\psi(x)-\psi(z)|\quad\text{for}\ \ x,z\in[0,1]$$

with a $c\in(0,\infty)$, then ξ is a.s. constant.

Proof. For every $n\in\mathbb{N}$ we have

$$f^n(x,\cdot)=\lambda f^{n-1}(x,\cdot)+\xi_n,$$

where

$$\xi_n(\omega_1,\omega_2,\ldots)=\xi(\omega_n)\quad\text{for}\ (\omega_1,\omega_2,\ldots)\in\Omega^\infty,$$

and

$$c\left|f^n(x,\omega)-f^{n-1}(x,\omega)\right|\le\left|\psi\big(f^n(x,\omega)\big)-\psi\big(f^{n-1}(x,\omega)\big)\right|\quad\text{for}\ \omega\in\Omega^\infty,$$

which implies that the sequence $\left(f^{n-1}(x,\cdot)+\frac{1}{\lambda-1}\xi_n\right)_{n\in\mathbb{N}}$ converges in probability to zero. Since

$$f^n(x,\cdot)+\frac{1}{\lambda-1}\xi_{n+1}=\lambda\Big(f^{n-1}(x,\cdot)+\frac{1}{\lambda-1}\xi_n\Big)+\frac{1}{\lambda-1}(\xi_{n+1}-\xi_n)$$

for $n\in\mathbb{N}$, it proves that the sequence $(\xi_{n+1}-\xi_n)_{n\in\mathbb{N}}$ converges in probability to zero. But $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables, the distribution of ξ_n is just the distribution of ξ for every $n\in\mathbb{N}$, whence (cf. [3, Theorem 9.1.3])

$$(\mu_\xi*\mu_{-\xi})\big((-\infty,-\epsilon]\cup[\epsilon,\infty)\big)=0\quad\text{for}\ \epsilon\in(0,\infty),$$

where μ_ξ and $\mu_{-\xi}$ denote the distributions of ξ and $-\xi$, respectively. Consequently,

$$(\mu_\xi*\mu_{-\xi})(\mathbb{R}\backslash\{0\})=0,$$

from which

$$1=(\mu_\xi*\mu_{-\xi})(\{0\})=\int_{\mathbb{R}}\mu_{-\xi}(\{-z\})\mu_\xi(dz)=\int_{\mathbb{R}}\mu_\xi(\{z\})\mu_\xi(dz),$$

and so ξ is a.s. constant. $\square$

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Karol Baron
Instytut Matematyki
Uniwersytet Śląski
Bankowa 14
40-007 Katowice
Poland
e-mail: baron@us.edu.pl

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